Université Libre de Bruxelles

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Project

Disjoint Compatible Perfect Matchings

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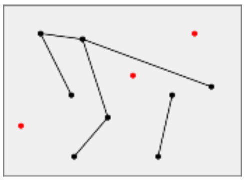
Introduction

This Web page presents some properties between two perfect matchings in planar straight line graphs. Below, a simple JavaScript application allows to experiment with these concepts.

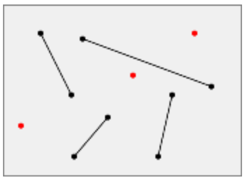
1 Some basic definitions

We will consider a particular form of ***undirected graph***, where each ***vertex*** is a ***point*** in the plane, and each ***edge*** is a ***segment*** (no curve) between two points. Moreover there is *no intersection*. We call that a ***planar straight line graph*** (***PSLG***).



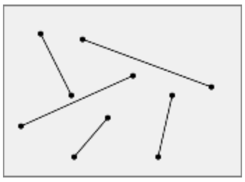
Planar straight line graph example.

A ***matching*** (or *independent edge set*) is a set of segments such that they have no point in common (i.e. if and only if each vertex has degree at most one). In this document we consider only matchings with ***no intersection***.



Matching example.
(If you click on this example or a following example, it will be loaded to the interactive application below.)

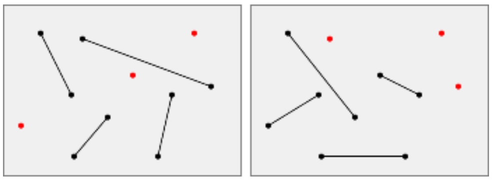
A matching is said ***perfect*** if and only if each point is an end-point of one and only one segment (i.e. if and only if each vertex has degree one). Of course, that requires an even number of *points*.



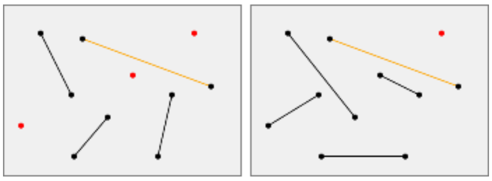
Perfect matching example.

A matching is said ***even*** if and only if it have a even number of *segments*. Else it said ***odd***.

Now we consider two matchings on a same PSLG. Two matchings are said ***disjoint*** if and only if their have no common segment.

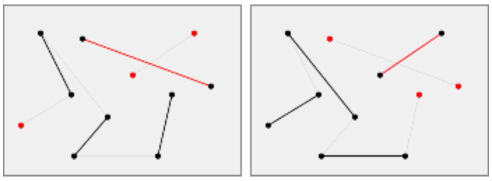


Disjoint matchings example.

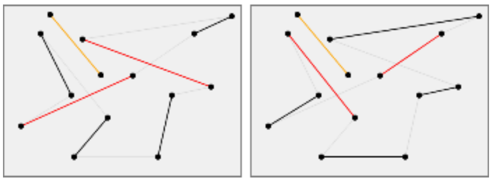


Not disjoint matchings example.
(Common segments are drawn in orange.)

Two matchings are said ***compatible*** if and only if their *union* is with no intersection.
Be careful, the *union* is the union of two sets (concept of set theory). And the *intersection* is the intersection of two segments (geometrical concept).

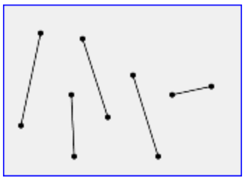


Not compatible matchings example.
(Segments such that the union intersect are drawn in red. And in thin silver are drawn segments of the other matching.)

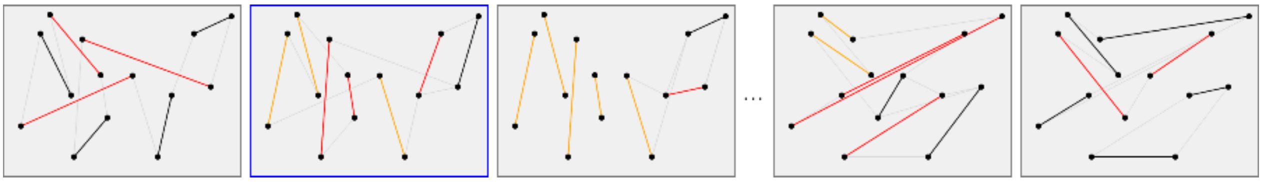


Not compatible *perfect* matchings example with one segment in common.

Let S a set of $2n$ points. If we called $p_1, p_2, p_3, \ldots, p_{2n}$ in increasing order of their x-coordinates (and for points with the same x-coordinate, in increasing order of their y-coordinates), the ***canonical perfect matching*** of S, written $N(S)$, is the perfect matching with segments $p_1 — p_2, p_3 — p_4, p_5 — p_6, \ldots p_{2n-1} — p_{2n}$.



Canonical perfect matching example.

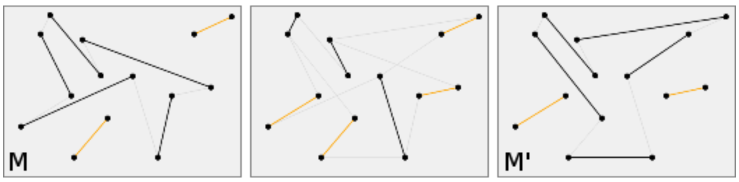


5 out of the 2736 possible perfect matchings for these set of points, where the second is the canonical perfect matching.

2 Transformation between two perfect matchings

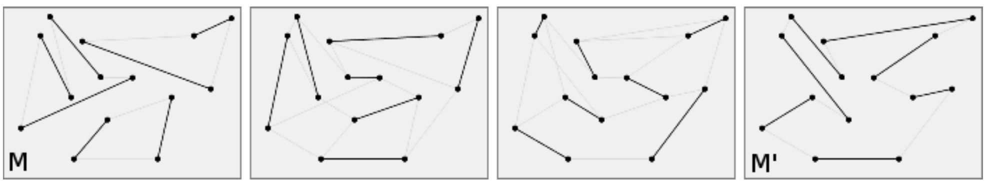
When two perfect matchings are *not* compatible, it may be possible to have a succession of perfect matchings two by two compatible, from the first one to the last one.

We call ***transformation*** between two *perfect* matchings a sequence of perfect matchings such that each consecutive pair of perfect matchings is compatible. More formally, with S a set of points, M and M' two perfect matchings of S, a transformation between M and M' of **length** k is a sequence $M = M_0, M_1, M_2, \ldots, M_k = M'$ of perfect matchings of S such that $\forall i \in \{0, 1, 2, \ldots, k-1\} : M_i$ and M_{i+1} are compatible.



Transformation example of length 2.

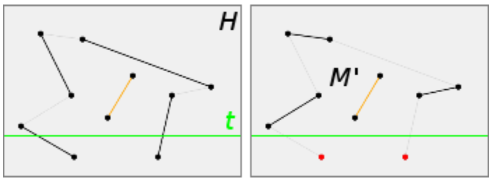
If moreover $\forall i \in \{0, 1, 2, \ldots, k-1\} : M_i$ and M_{i+1} are disjoint, the transformation is called a ***disjoint transformation***.



Disjoint transformation example of length 3.

2.1 Lemmas

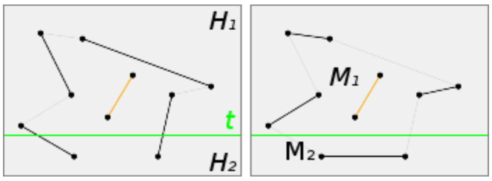
> ***Lemma i.***
> $\forall$ perfect matching M, $\forall$ line t cutting an even number of segments of M (such that t contains no vertex),
> let H the halfplane determined by t, let S the set of vertices of M in H,
> $\exists$ perfect matching M' of S : M and M' are compatible



Lemma i example.

The article [1] p. 7 contains two proofs of this lemma, with two different ideas (the concept of segment extensions or thicken segments to infinitesimal triangles).

> ***Lemma ii.***
> $\forall$ perfect matching M, $\forall$ line t cutting an even number of segments of M (such that t contains no vertex),
> let halfplanes H_1 and H_2 determined by t, let S_1 and S_2 sets of vertices of M in H_1 and in H_2,
> $\exists$ perfect matchings M_1 of S_1 and M_2 of S_2 : M and $(M_1 \cup M_2)$ are compatible



Lemma ii example.

Proof.
By two applications of lemma i, there are perfect matchings M_1 of S_1 and M_2 of S_2 such that M and M_1 are compatible, and M and M_2 are compatible. Since M_1 and M_2 are separated, then $M_1 \cup M_2$ is also a perfect matching and M and $(M_1 \cup M_2)$ are compatible.

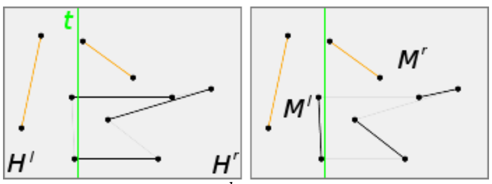
□

> ***Lemma iii.*** $\forall S$ of $2n$ points, $\forall$ perfect matchings M of S, $\exists$ transformation of length at most $\lceil \lg(n) \rceil$ between M and $N(S)$

Proof by induction on n.

- Base case: $n = 1$. All perfect matchings with 2 points are canonical, so the transformation has length $0 = \lg(1)$.
- Inductive hypothesis. Assume that the lemma is true for all value less than n with $n > 1$.
- Inductive step.
  Let a vertical line t cutting the plane in two parts (left H^l and right H^r) such that H^l contains a set S^l of $2\lfloor \frac{n}{2} \rfloor$ points and H^r contains a set S^r of $2\lceil \frac{n}{2} \rceil$ points.

  Let m the number of segments cut. The subset of segments in the left that no intersects with t is a perfect matching, so $2\lfloor \frac{n}{2} \rfloor - m$ is even and m too.

  Example with $n = 5$: $|S^l| = 4$ and $|S^r| = 6$, $m = 2$

  By lemma ii, there is perfect matchings M^l of S^l and M^r of S^r such that M and $(M^l \cup M^r)$ are compatible

  By inductive hypothesis to M^l and M^r there are theses transformations:
  $M^l = M^l_0, M^l_1, M^l_2, \ldots, M^l_k = N(S^l)$
  $M^r = M^r_0, M^r_1, M^r_2, \ldots, M^r_k = N(S^r)$
  where $\lceil \lg \lfloor \frac{n}{2} \rfloor \rceil \leq \lceil \lg \lceil \frac{n}{2} \rceil \rceil \leq \lceil \lg(n) \rceil - 1 = k$

$\forall i : M_i^l$ and M_{i+1}^l are compatible, and M_i^r and M_{i+1}^r are compatible.

Let $M_i = M_i^l \cup M_i^r$. It is a perfect matching of S, because M_i^l and M_i^r are separated by t. And M_i and M_{i+1} are compatible.

$N(S) = N(S^l) \cup N(S^r) = M_k$
so $M, M_0, M_1, M_2, \ldots, M_k$ is a transformation between M and $N(S)$ of length $\lg(n)$

□

2.2 Theorem

Theorem iv. $\forall$ perfect matchings M and M', $\exists$ transformation of length at most $2\lceil \lg(n) \rceil$ between M and M'

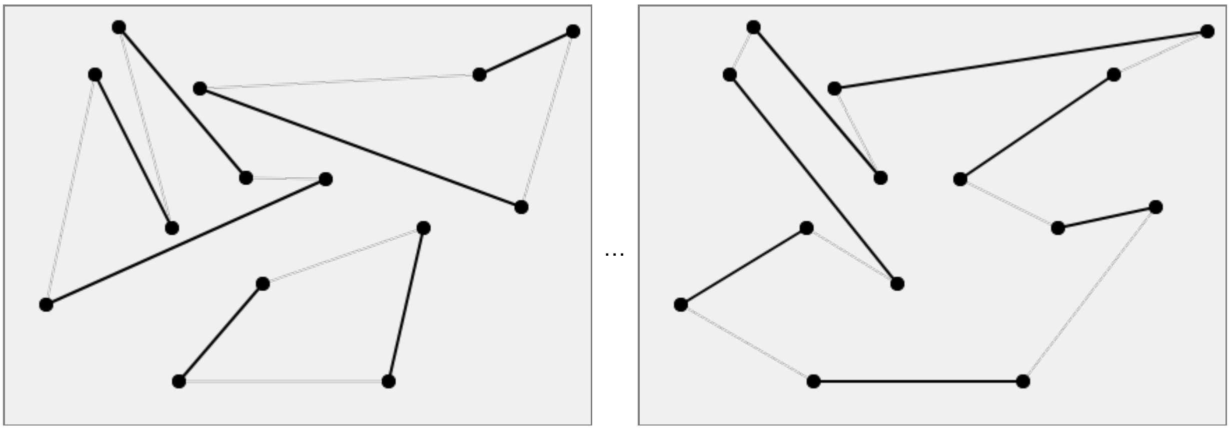
Proof.
Let S the set of $2n$ points. By lemma iii there are perfect matchings M and M' such that
$M = M_0, M_1, M_2, \ldots, M_k = N(S)$ and
$M' = M'_0, M'_1, M'_2, \ldots, M'_{k'} = N(S)$ with $k, k' \leq \lceil \lg(n) \rceil$.
Thus $M_0, M_1, M_2, \ldots, M_k = M'_{k'}, \ldots, M'_2, M'_1, M'_0 = M'$ is a transformation of length at most $2\lceil \lg(n) \rceil$.

□

3 Experiment by yourself

3.1 Interactive application

Click in the two zones to add/remove point or segment (only one segment by point and with no intersection). You can also click on one matching example to load it in the interactive zones. Wait a moment above a button to bring up a short explanation tooltip.

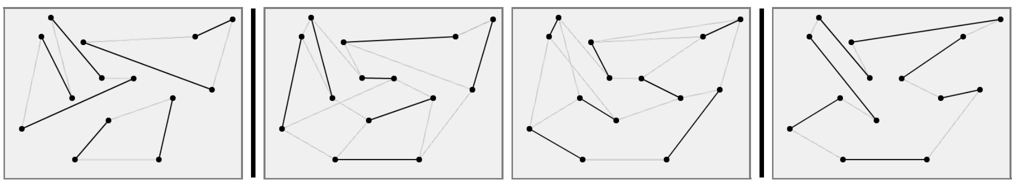


7 segments	Even? ✗	
Perfect? ✔	Canonical? ✗	Vertical-horizontal? ✗

7 segments	Even? ✗	
Perfect? ✔	Canonical? ✗	Vertical-horizontal? ✗

14 points ✔ 2736 possible perfect matching(s) ≲ 135135 Minimal length transformation ≲ 8
Different? ✔ Disjoint? ✗ (common segments)
Compatible? ✗ (intersect segments)

4 matchings:



3.2 Some explanations

You can add or remove points one by one, but only *perfect matchings* are useful. Isolated points are drawing in red. To remove a point, click on it. When a point is removed, all its segment are removed.

To add a segment, click to add the first point, then click to add the second point and the segment. It is impossible to add a segment to a point that already belongs to a segment, or to add a segment that intersects with an other segment. To remove a segment (and keeping its points), click to a first point and then to the second point.

If a matching is a *canonical perfect matching* then it is in a blue frame.

For two consecutive matchings in the list, common segments are in orange and segments that intersects between them are in red.

By default, all segments of the immediately previous and next matchings are drawn in thin silver. An option permits to draw all segments that are in at least one matching in the list, or on the contrary to disable that and to draw only the segments of the matching.

For the matchings of the two interactive zones, some properties are displayed. Below that, some global properties between these two matchings are also displayed.

The matching of the left interactive zone is also displayed in the first position in the list. And the matching of the right interactive zone is displayed in the last position. Between them, are displayed all other perfect matchings or the intermediary perfect matchings to be a transformation.

Build all perfect matchings or build a shortest transformation are *very expensive operations*, so they are impossible with a lot of points. Moreover only transformations of length 2 and 3 are tried.

The upper bound given by the application for the number of possible perfect matchings is very rough. In fact it is the number of *general* (with possible intersection) perfect matchings.

With $n = 2k$: $|\{\text{general perfect matching}\}| = (n-1).(n-3).(n-5)\ldots5.3 = \frac{n!}{n.(n-2).(n-4)\ldots4.2} = \frac{n!}{2^k.k!}$

And the STIRLING's approximation $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$ give the approximation $\sqrt{2}(\frac{n}{e})^k$.

3.3 Sources code

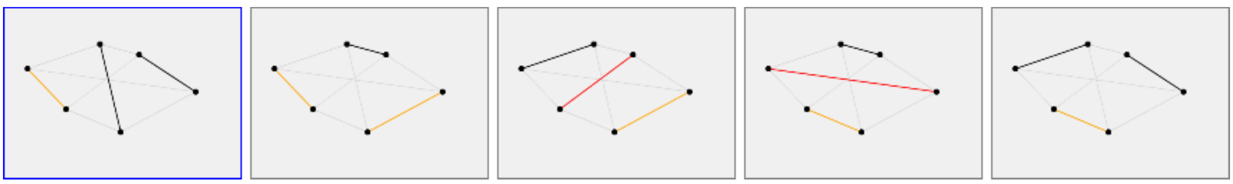
The interactive application was tested on *Firefox* 52.0 (and a little on *Chromium* 57.0) (requires ECMAScript/JavaScript 2015 6$^{\text{th}}$ and modern browser).

- The online JavaScript **source codes/** for this application. License GPLv3
- The complete sources of this project on **Bitbucket**.
- **Online HTML documentation** of source codes.

4 Compatible matching conjecture

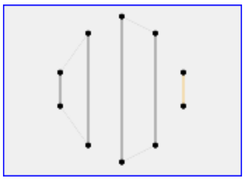
> ***Compatible matching conjecture.*** $\forall$ *even* perfect matching M, $\exists$ perfect matching M' : M and M' are disjoint and compatible

First observation: the *even* condition is necessary. Indeed, it is easy to find simple odd perfect matching M such that doesn't exist a disjoint and compatible perfect matching M'. In the example below, for the first perfect matching with 3 segments we see that none of four other perfect matchings is appropriate. (In fact for this example, there is only one possible pair (M, M'), the 2$^{\text{nd}}$ and the 5$^{\text{th}}$ matchings.)



Odd perfect matching example, with no disjoint and compatible other matching.

The article [1] p. 9 explains how construct odd perfect matching with arbitrary size with no disjoint and compatible other matching. The idea is to consider the canonical perfect matching on points disposed on a circle like this:



Odd perfect matching example disposed on a circle, with no disjoint and compatible other matching.

This conjecture was **proved** in the article [3].

References

[1] Oswin AICHHOLZER, Sergey BEREG, Adrian DUMITRESCU, Alfredo GARCÍA, Clemens HUEMER, Ferran HURTADO, Mikio KANO, Alberto MÁRQUEZ, David RAPPAPORT, Shakhar SMORODINSKY, Diane L. SOUVAINE, Jorge URRUTIA, David R. WOOD.
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[3] Mashhood ISHAQUE, Diane L. SOUVAINE, Csaba D. TÓOTH.
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In Proceedings of SOCG, pp. 125-134, New York, NY, USA, 2011. ACM

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- JavaScript. Mozilla Developer Network
- Plug-ins: MathJax, ViewerJS
- Wolfram MathWorld

Index

- $N(S)$
- canonical perfect matching
- compatible
- Compatible matching conjecture
- disjoint
- disjoint transformation
- edge
- even
- matching
- odd
- perfect
- planar straight line graph
- point
- PSLG
- segment
- transformation
- undirected graph
- vertex

Contents

Introduction

1 Some basic definitions

2 Transformation between two perfect matchings
2.1 Lemmas
2.2 Theorem

3 Experiment by yourself
3.1 Interactive application
3.2 Some explanations
3.3 Sources code

4 Compatible matching conjecture

References

Index